Generic measure for the quantum correlation of the two-qubit systems: the average of the spin-correlation elements

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Based on the Pauli spin operators we develop the notion of the spin-correlation matrix for the two-qubit system. If this matrix is non-zero, the measure of the correlation between the qubits is the average of the non-zero elements. Trivially, for zero matrix the bipartite is uncorrelated. This criterion turns out to be a necessary and sufficient condition for the full correlation, where it includes information on both entanglement and correlation other than entanglement. Moreover, we discuss to what extent this criterion can give information on the entanglement of the system. The criterion is generic in the sense that it can be applied to mixed and pure systems. Also, it can be easily extended to treat the correlation of multipartite systems. We compare the results obtained from this criterion to those from concurrence for various examples and we gain agreement regarding entanglement. We believe that this criterion may have a wide range of potential applications in quantum information theory.

Quantum entanglement is a useful physical resource for quantum information processing. For instance, the predicted capabilities of a quantum computer rely crucially on entanglement [1], and a proposed quantum cryptographic scheme achieves security by converting

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shared entanglement into a shared secret key [2]. For both theoretical and potentially practical reasons, it has become essential to quantify entanglement in physical systems. Thereby, this subject has attracted massive interest recently. Over the years, a number of different criteria for quantifying entanglement of physical systems have been developed such as negativity of the partial transpose [3], concurrence [4, 5], correlation functions [6–8] and quantum discord [9, 10]. However, each one of these criteria has advantages and disadvantages.

In this Letter, we propose a simple and novel correlation criterion for the two-qubit system. The criterion includes information on both entanglement and correlation other than entanglement. Throughout this letter, for the entanglement we use the phrase "nonclassical correlation", however, for the other correlation we metaphorically use "classical correlation". The criterion is generic in the sense that it works for both mixed and pure systems. Also, we argue how one can extend it to assess the correlation of the multipartite system. Moreover, we discuss to what extent the information given by this criterion represents the entanglement of the system. The steps for getting the criterion can be described as follows. We construct the spin-correlation matrix by calculating the correlation elements of all pairs of the Pauli spin operators of the qubits (i.e. $\hat{\sigma}_x^{(j)}, \hat{\sigma}_y^{(j)}, \hat{\sigma}_z^{(j)}$). If this matrix is non-zero, the proposed criterion, which can give complete information on the correlation of the bipartite, is the average of the non-zero elements. Nonetheless, if all the elements of the matrix are zero, the average will be automatically zero declaring uncorrelation. We call this criterion the correlation indicator and denote it by I. It is worth mentioning that, in some contexts, the concept of the correlation function has already been considered in the literature [6–8]. We illustrate that $I \neq 0$ is a necessary and sufficient condition for the bipartite exhibiting correlation. Needless to say that if entanglement exists, it will be a subset of this correlation. Furthermore, we compare the results obtained from our criterion with those from the concurrence $C(\rho)$ for various examples. The main objective of this comparison is to show that when $C(\rho)$ exhibits entanglement, I provides quite similar information. Besides, I can include additional information on the classical correlation. It is valuable to mention the recent discovery that correlations other than entanglement can be responsible for the quantum computational efficiency of the deterministic quantum computation with one pure qubit [11]. Furthermore, the comparison allows us to know when the information given by I represents the entanglement. Precisely, we have noted that when $I > 1/3$ the bipartite is entangled, however, for $I \leq 1/3$ the bipartite may be correlated classically or nonclassically.

The limit 1/3 is obtained from the comparison with the entanglement of the Werner state. As we proceed, we shed some light on the sudden-death of entanglement phenomenon (SDE) [12–17] in terms of I. The SDE means that the entanglement in the system disappears in a finite interaction time, in contrast to the exponential decay of the excited states (or coherence) of some systems. Despite the fact that the SDE sounds mysterious, there is no dynamical distinction between separable and entangled states, since quantum states can, in general, evolve back and forth across the boundary between separable and entangled states [9, 10]. We show that I cannot detect the SDE. In this respect its behavior is quite similar to that of the quantum discord [9, 10]. We will describe all these findings in detail below.

Our starting point is the generic density matrix of the bipartite, which in the standard basis $\{|e_1, e_2\rangle, |e_1, g_2\rangle, |g_1, e_2\rangle, |g_1, g_2\rangle\}$ has the form:

$$\hat{\rho} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \tag{1}$$

where $b_{11} + b_{22} + b_{33} + b_{44} = 1$.

Our main concern in this Letter is the evaluation of the full correlation inherited in the quantum system. Based on that a state of a multipartite quantum system is called correlated if it cannot be factorized into individual states. Mathematically, the quantum system described by $\hat{\rho}$ is correlated iff it cannot be expressed as:

$$\hat{\rho} = \bigotimes_{j=1}^{n} \hat{\rho}_j, \tag{2}$$

where $\hat{\rho}_j$ is the density matrix of the jth-party and n is the number of parties. The new criterion is working under the condition that Eq. (2) is satisfied regardless of the system being in the pure or in the mixed state.

Next, we shed some light on the concurrence, which will be the benchmark for the suggested criterion regarding entanglement. The concurrence $C(\rho)$ can be described as follows [4, 5]:

$$C(\rho) = \max\{0, \sqrt{\nu_1} - \sqrt{\nu_2} - \sqrt{\nu_3} - \sqrt{\nu_4}\}, \tag{3}$$

where the elements ν_j are the eigenvalues in decreasing order of the auxiliary matrix

$$\xi = \hat{\rho}(\hat{\sigma}_y \otimes \hat{\sigma}_y)\hat{\rho}^*(\hat{\sigma}_y \otimes \hat{\sigma}_y), \tag{4}$$

where $\hat{\rho}^*$ denotes the complex conjugation of $\hat{\rho}$ in the standard basis and $\hat{\sigma}_y$ is the Pauli-spin matrix. We have $1 \geq C(\rho) \geq 0$, where $C(\rho) = 0$ indicates separability (zero entanglement) and $C(\rho) = 1$ means maximal entanglement.

Now, we are in a position to develop the new criterion, which quantifies the correlation of the bipartite. For this purpose, we define the spin-correlation matrix as:

$$I_{xyz} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, \tag{5}$$

where

$$I_{ij} = |\langle \hat{\sigma}_i^{(1)} \hat{\sigma}_j^{(2)} \rangle| - |\langle \hat{\sigma}_i^{(1)} \rangle \langle \hat{\sigma}_j^{(2)} \rangle|, \quad i,j = x, y, z. \tag{6}$$

The modulus in (6) is given to achieve a quantitative description for the correlation of the bipartite. In this regard, the elements I_{ij} are quite different from the covariance ones given in [7]. The reason for using all possible elements I_{ij} in the matrix (5) is that the occurrence of correlation (and consequently entanglement) at a time instant t necessarily implies the existence of at least a couple of operators $\hat{\sigma}_i^{(j')}$ acting on the bidimensional Hilbert spaces of spin 1 and 2 such that the corresponding element is non-zero. When $I_{ij} \neq 0, i \neq j$ or $I_{ii} \neq 0$ means that the bipartite is correlated in the ij-plane or in the i-axis direction. Actually, there is a similarity between the structure of the matrix (5) and that of the moment of inertia tensor matrix of the 3 dimensional rigid body as well as that of the covariance matrix of a trivariate random vector whose probability density function is proportional to the pointwise density of the rigid body. For any arbitrary quantum atomic state, one can easily verify that $|\langle \hat{\sigma}_i^{(1)} \hat{\sigma}_j^{(2)} \rangle| \geq |\langle \hat{\sigma}_i^{(1)} \rangle \langle \hat{\sigma}_j^{(2)} \rangle|$ for all i, j. Also, the modulus of the expectation value of any multiples of the Pauli spin operators always exists inside or on the surface of the Bloch sphere. Thereby, we have $0 \leq I_{ij} \leq 1$, where $I_{ij} = 0$ (1) means that the bipartite is disentangled (maximally correlated/entangled) in the ij-plane. Moving forward, we show that the matrix (5) includes all necessary information about the correlation/entanglement of the bipartite. The elements of the density matrix (1) can be categorized into three groups, namely, diagonal, anti-diagonal $\{b_{14}, b_{23}, b_{32}, b_{41}\}$, and the remaining elements. The first group contributes to the element I_{zz}, the second one gives information on $\langle \hat{\sigma}_i^{(1)} \hat{\sigma}_i^{(2)} \rangle$ with $i = x, y$, while the last group provides information on $\langle \hat{\sigma}_i^{(1)} \hat{\sigma}_j^{(2)} \rangle$ and $\langle \hat{\sigma}_l^{(k)} \rangle$ where $i, j = x, y, z, i \neq j, \quad l = x, y$ and $k = 1, 2$. In order to obtain full information on the

correlation of the bipartite, one has to deal with all elements in the matrix (5), not only some of them. This contrasts with the early study in [7], where the authors confined themselves to two elements only. We elaborate this point by giving the following example. Assume that we would like to assess the correlation of the of the quasi-Bell-state:

$$|\Psi\rangle = \cos\theta|e_1, e_2\rangle + \sin\theta|g_1, g_2\rangle. \tag{7}$$

If we restrict ourselves to the elements $I_{i \neq j}$, we will find that the bipartite is disentangled, which is not true. Considering all these facts, the new criterion (, i.e. the correlation indicator I), which can give accurate information on the correlation (classical or nonclassical) of the bipartite, is:

$$I = \frac{1}{N} \sum_{i,j=x,y,z} I_{ij}, \tag{8}$$

where N represents the number of the non-zero elements in I_{xyz}. As we mentioned above, if $I_{ij} = 0$ for all i and j, then $I = 0$ and the bipartite is disentangled. The indicator I ranges between 0 (disentangled) and 1 (maximally correlated/entangled) bipartite. It is obvious that I has various merits. It is a well-behaved quantity and can be applied for both pure and mixed systems. It is easy to be manipulated since it deals with the moments of the spin operators. It gives a simple means to interpret and to measure the correlation. Furthermore, the criterion (8) can be easily extended to treat the correlation/entanglement of the multipartite systems.

Our next task is to check the efficiency of the criterion (8). This may be done by applying it to some examples and comparing the obtained results with those from the concurrence. To achieve this goal we give five examples; two of them (examples 4 and 5) are devoted to the SDE.

Example (1): Assume we would like to quantify the correlation between two qubits of the three-particle Greenberger-Horne-Zeilinger state (GHZ) [18], which has the form, e.g.,:

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}[|e_1, e_2, e_3\rangle + |g_1, g_2, g_3\rangle]. \tag{9}$$

The density matrix of any two qubits of the state (9), say 1, 2, can be evaluated by tracing over the third particle as:

$$\hat{\rho} = \frac{1}{2}[|e_1, e_2\rangle\langle e_2, e_1| + |g_1, g_2\rangle\langle g_2, g_1|]. \tag{10}$$

Foremost, the density matrix (10) represents a maximally mixed state. In terms of the concurrence, state (10) is disentangled, where $C(\rho) = 0$. This is related to the definition of $C(\rho)$, which is based on the spin flip transformation. Nevertheless, state (10) is maximally correlated with respect to our criterion, where $I = 1$.

Example (2): We study the entanglement of the quasi-Bell-state (7). For this state, the indicator I can be expressed as:

$$I = \frac{1}{3}(I_{xx} + I_{yy} + I_{zz}) = \frac{1}{3}[\sin^2(2\theta) + 2|\sin(2\theta)|]. \tag{11}$$

Also, its concurrence takes the form:

$$C(\rho) = \sin^2(2\theta). \tag{12}$$

It is evident from equations (11) and (12) that both I and $C(\rho)$ have almost the same behavior. In this case, I gives information on the entanglement.

Example (3). Here we shed some light on correlation of the general case of 2-q-bit pairs. Also we compare the results obtained from our criterion with those given in [19, 20] as well as with the concurrence. To this end, we use the general form of a 2-q-bit state [19, 20]:

$$\widetilde{\rho} = \frac{1}{4}[1 + \overrightarrow{\sigma}.s^{\downarrow} + \overrightarrow{t}.\tau^{\downarrow} + \overrightarrow{\sigma}.\overrightarrow{\downarrow C}.\tau^{\downarrow}], \tag{13}$$

where $\overrightarrow{\downarrow C} = \langle \sigma^{\downarrow}\overrightarrow{\tau} \rangle$. We employ the terminology and the notational conventions of [19, 20], in which the individual q-bits are described by means of analogs of Pauli's spin vector operator: $\overrightarrow{\sigma}$ for the first q-bit, $\overrightarrow{\tau}$ for the second. The arrows $\rightarrow, \downarrow, \overrightarrow{\downarrow}$ denote row vector, column vector and the cross dyadic. In [19, 20] it has been shown that the characterization of the 2-q-bit states produced by some source requires the experimental determination of 15 real parameters. Based on the knowledge of these parameters, the 2-q-bit state has been classified into six classes of families of locally equivalent states. Simple criteria have been stated for checking a given states positivity and separability through fulfilling certain inequalities. The technique has been employed in finding the Lewenstein-Sanpera decompositions (LS) [21] for the state $\widetilde{\rho}$. This will be our main concern in this comparison. Precisely, LS states that any 2-q-bit state $\widetilde{\rho}$ can be written as a mixture of a separable state $\hat{\rho}_{sep}$ and a non-separable pure state $\hat{\rho}_{pure}$:

$$\widetilde{\rho} = \lambda\hat{\rho}_{sep} + (\lambda - 1)\hat{\rho}_{pure}. \tag{14}$$

There are many different such LS decompositions with varying values of λ. Among them is the unique optimal decomposition, the one with the largest λ value,

$$\widetilde{\rho} = \mu \hat{\rho}_{sep}^{(opt)} + (1 - \mu)\hat{\rho}_{pure}^{(opt)}, \quad \mu = \max\{\lambda\}. \tag{15}$$

In this case μ represents the degree of separability of $\widetilde{\rho}$; the value $\mu = 0$ obtains only if $\widetilde{\rho}$ itself is a nonseparable pure state. For the separability of (13), we restrict ourselves to two examples, which have been studied in detail in [19, 20], namely, Werner states and the rank-2 states.

For Werner states, the generic density matrix (13) takes the form:

$$\hat{\rho}_w = (1 - x)\hat{\rho}_{chaos} + x\hat{\rho}_{Bell} = \frac{1}{4}[1 - x\overrightarrow{\sigma}.\sum_{k=1}^{3} e_k^{\downarrow}\overrightarrow{n}_k.\tau^{\downarrow}], \tag{16}$$

where the subscripts *choas* and *Bell* stand for the 2-qubit density matrices of the thermal and the Bell state, respectively. For the sake of clarification, in the transition from (13) to (16) one has $s^{\downarrow} = 0$, $\overrightarrow{t} = 0$, $\overset{\downarrow}{\overrightarrow{C}} = -x\sum_{k=1}^{3} e_k^{\downarrow}\overrightarrow{n}_k$. For this state, the degree of separability μ [19], concurrence $C(\rho)$ and correlation indicator I for (16) can be expressed as:

$$\mu = \begin{cases} 1 & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{3}, \\ \frac{3}{2}(1 - x) & \text{if } \frac{1}{3} < x \leq 1, \end{cases} \quad C(\rho) = \max\{0, \frac{3}{2}(x - \frac{1}{3})\}, \quad I = |x|. \tag{17}$$

The result of μ agrees well with both the numerical findings of Lewenstein and Sanpera [21] and the concurrence. Moreover, it is clear that $\mu, C(\rho)$ and I agree with each other for the extreme values. It should be borne in mind that I has been derived based on the definition (2). Based on that, the correlation found in the interval $-\frac{1}{3} \leq x \leq \frac{1}{3}$ in I can be understood classically. Inspired by this observation we can modify I, for this case, to provide the nonclassical correlation in $\frac{1}{3} < x \leq 1$ by subtracting $1/3$, i.e. $I - 1/3$. However, the modified version needs to be normalized to unity at $x = 1$, as it should be. Then we get $\widetilde{I} = C(\rho)$. It is evident that the degree of separability μ and $\widetilde{I}$ give identical information on the entanglement of $\hat{\rho}_w$. Also, one can conclude: for any bipartite when $I > 1/3$ the correlation can be interpreted as entanglement.

Now we give attention to states of rank 2 [20]:

$$\hat{\rho}_{rank2} = \frac{1}{4}[1 + (\sigma_3 + x\tau_3)\sin(\theta) + (\sigma_1\tau_1 - x\sigma_2\tau_2)\cos(\theta) + x\sigma_3\tau_3], \tag{18}$$

with $-1 < x < 1$. The generic form for states of rank 2 can be found in [19]. The degree of separability, concurrence and correlation indicator I for (18) are:

$$\mu = \begin{cases} 1 & \text{if } \cos(\theta) = 0, \\ 1 - |x| & \text{if } \cos(\theta) \neq 0, \end{cases} \qquad C(\rho) = |x \cos(\theta)|, \qquad (19)$$

$$I = \tfrac{1}{3}[(1 + |x|)|\cos(\theta)| + |x| \cos^2(\theta)].$$

It is obvious that these three criteria provide the same information for the cases $\cos(\theta) = 0$ and $x = \cos(\theta) = 1$. However, for $x = 0, \cos(\theta) = 1$, the bipartite is separable in terms of μ and $C(\rho)$, but $I = 1/3$. This confirms the above observation: the correlation exhibited by I is nonclassical whenever $I > 1/3$. For certain range of the parameters, $C(\rho)$ and I have quite similar behavior, but this is not the case for μ. This is noticeable by comparing different expressions in (19). As an example, when $x = 1, \cos(\theta) = 1/2$ we have $\mu = 0$ (maximum entanglement), whereas $C(\rho) = 1/2$ and $I = 5/12$.

Example (4): This example and the following one have been very often used in the literature to establish the concept of the SDE [12–17]. Thus, we use them to discuss the SDE in the framework of I. We consider the evolution of the state (7) with the two-mode Jaynes-Cummings model [13–17], which may be described by the interaction Hamiltonian:

$$\hat{H}_I = \lambda \sum_{j=1}^{2} (\sigma_+^{(j)} \hat{a}_j + \sigma_-^{(j)} \hat{a}_j^\dagger), \qquad (20)$$

where $\sigma_+^{(j)}$ $(\hat{a}_j^\dagger)$ and $\sigma_-^{(j)}$ $(\hat{a}_j)$ are the raising and the lowering atomic (field) operator of the jth-party, respectively, λ is the coupling constant and we take $\hbar = 1$. For the initial field state $|0_1, 0_2\rangle$, the elements of the matrix (1) of the two qubits take the forms [16]: $b_{11} = \cos^2 \theta \cos^4 T$, $b_{22} = b_{33} = \tfrac{1}{4} \cos^2 \theta \sin^2(2T)$, $b_{44} = \cos^2 \theta \sin^4 T + \sin^2 \theta$, $b_{14} = b_{41} = \tfrac{1}{2} \sin(2\theta) \cos^2 T$, with $T = t\lambda$, and the other elements are zero. The concurrence and the indicator I of the two qubits of this system read:

$$C(\rho) = 2 \cos^2 T \cos^2 \theta \quad \max\{0, |\tan \theta| - \sin^2 T\}, \qquad (21)$$

$$I = \tfrac{1}{3}\{1 - \sin^2(2T) \cos^2 \theta - [\cos(2T) \cos^2 \theta - \sin^2 \theta]^2$$
$$+ |\sin(2\theta) \cos^2 T + \tfrac{1}{2} \cos^2 \theta \sin^2(2T)| + |\sin(2\theta) \cos^2 T - \tfrac{1}{2} \cos^2 \theta \sin^2(2T)|\}. \qquad (22)$$

Comparison between (21) and (22) reveals that $C(\rho)$ exhibits SDE only when $|\tan \theta| \leq \sin^2 T$, while I cannot. The reason is that I includes the classical and the nonclassical

correlation, while $C(\rho)$ has the nonclassical one. In Figs. 1(a) and (b) we depict I and $C(\rho)$ for two different cases $\theta = \pi/4$ and $\pi/6$, respectively. From Fig. 1(a), the behavior of the two criteria is almost the same, where, for this case, $C(\rho) = \cos^4 T$ and $I = \frac{1}{3}(2 + \cos^2 T)\cos^2 T$. Fig. 1(b) shows that I has smooth oscillatory behavior without SDE, whereas $C(\rho)$ non-smoothly becomes and stays zero for a finite interval of time. As a result of the lossless nature of the evolution, the original entanglement value occurs in a periodic way following each sudden death event. The beauty of this example is that SDE occurs without decoherence of the traditional type. The origin of the occurrence of SDE in $C(\rho)$ is in the cut-off existing in the definition of the concurrence (3). As we proceed, one can observe that I vanishes only at discrete instants, i.e. $T = \pi/2$, declaring the disentanglement in the bipartite. One can verify this result by substituting $(T, \theta) = (\pi/2, \pi/6)$ into the density matrix of the bipartite, which reduces to $\hat{\rho} = |g_1\rangle\langle g_1| \bigotimes |g_2\rangle\langle g_2|$, i.e. the bipartite becomes pure and separable. In other words, $I = 0$ means that the information is locally accessible and can be obtained by distant independent observers without perturbing the bipartite state. The horizontal dashed line in Fig. 1(b) confirms our observation: for $I > 1/3$ the bipartite is entangled.
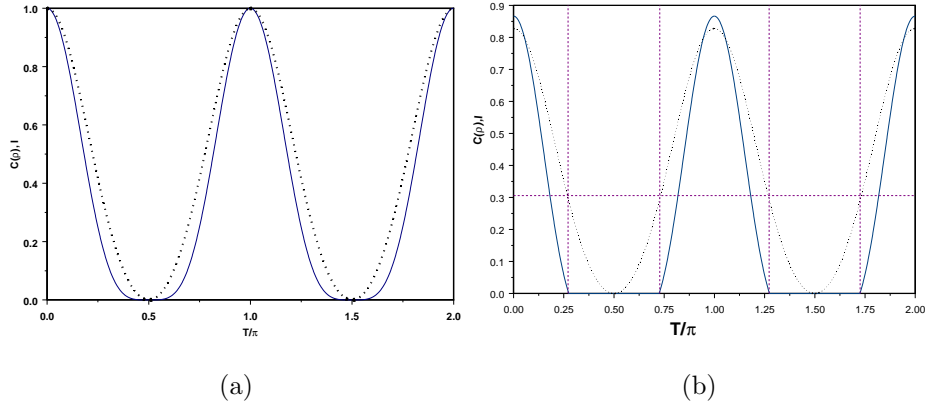

(a) (b)

FIG. 1: Atom-atom entanglement quantified by $C(\rho)$ (solid curve) and I (dashed curve) when the atoms are initially prepared in state (7) with $\theta = \pi/4$ (a) and $\pi/6$ (b). The vertical and horizontal dashed lines in (b) to show the interval of the classical correlation in I or the the segment of SDE in concurrence. The time scale is the vacuum Rabi period.

Example (5): Here, we consider two two-level atoms coupled individually to two cavities which are initially in their vacuum states. In the framework of system-plus-environment, the two two-level atoms are identified as the system of interest, whereas the two cavities

serve as the environments. The total Hamiltonian describing this scenario is [13]:

$$\hat{H} = \frac{\omega_1}{2}\sigma_z^{(1)} + \frac{\omega_2}{2}\sigma_z^{(2)} + \sum_k \left[w_k^{(1)}\hat{a}_k^\dagger \hat{a}_k + w_k^{(2)}\hat{b}_k^\dagger \hat{b}_k \right]$$

$$+ \sum_k \left\{ \left[g_k \sigma_+^{(1)}\hat{a}_k + g_k^* \sigma_-^{(1)}\hat{a}_k^\dagger \right] + \left[f_k \sigma_+^{(2)}\hat{b}_k + f_k^* \sigma_-^{(2)}\hat{b}_k^\dagger \right] \right\}, \tag{23}$$

where g_k, f_k are coupling constants and the other notations have the same standard meaning. When the atoms are initially entangled with each other but not with the cavities, the solution of the system has been derived already in [13] via the Born-Markov approximation. We focus our attention on the case that the atoms are initially in the mixed entangled state with $b_{11} = a/3 \geq 0, b_{22} = b_{33} = b_{23} = b_{32} = 1/3, \quad b_{44} = (1-a)/3$ and the other elements are zeros. Therefore, the dynamical density matrix elements of the two atoms have the form [13]:

$$b_{11}(t) = \tfrac{1}{3}\gamma_1^2\gamma_2^2 a, \quad b_{22}(t) = \tfrac{1}{3}(\gamma_1^2 + \gamma_1^2\Omega_2^2 a), \qquad b_{33}(t) = \tfrac{1}{3}(\gamma_2^2 + \gamma_2^2\Omega_1^2 a),$$

$$b_{44}(t) = \tfrac{1}{3}(1 - a + \Omega_1^2 + \Omega_2^2 + \Omega_1^2\Omega_2^2 a), \quad b_{23}(t) = b_{32}(t) = \tfrac{1}{3}\gamma_1\gamma_2, \tag{24}$$

and the other elements are zero. In using the Markov limit results and assuming the cavities are similar, we have $\gamma_1 = \gamma_2 = \exp(-\Gamma t/2), \quad \Omega_1 = \Omega_2 = \sqrt{1 - \exp(-\Gamma t)}$, where the Γ is the Einstein A coefficient for the two-level atoms in the cavities. The concurrence and the correlation indicator of this case are [13]:

$$C(\rho(T)) = \max\{0, |b_{23}(T)| - \sqrt{b_{11}(T)b_{44}(T)}\},$$
$$I(T) = \tfrac{4}{3}b_{23}(T) + \tfrac{1}{3}[1 - 4b_{22}(T)] - \tfrac{1}{3}\left[1 - 2b_{11}(T) - 2b_{22}(T)\right]^2, \tag{25}$$

where $T = \Gamma t$. For the initial state the expressions (25) reduce to

$$C(\rho(0)) = \frac{2}{3}[1 - \sqrt{a(1-a)}], \quad I(0) = \frac{2}{27}(2a - 4a^2 + 7). \tag{26}$$

It is evident that the initial two qubits are always entangled in terms of the two criteria (where $I > 1/3$) regardless of the values of a. In Fig. 2(a) and (b) we have plotted $C(\rho(T))$ and $I(T)$, respectively. Fig. 2(a) presents the well-known figure of the SDE. This figure shows that within the general exponential character evident in (24), disentanglement can be completed, i.e. $C(\rho(T)) = 0$, in a finite time while the local decoherences need an infinite time. For instance, when $a = 1$, this finite time is $T = \ln(\frac{2+\sqrt{2}}{2})$. The comparison between figure 2(a) and 2(b) reveals when $C(\rho(T))$ exhibits entanglement, $I(T) \neq 0$ declaring that

the bipartite is correlated. The behavior of the $C(\rho(T))$ and $I(T)$ are quite different. For $I(T)$, at $T = 0$, $I(T)$ exhibits its maximum value as given by (26). As the interaction time increases, $I(T)$ goes down rapidly until local minima are achieved, which result from the property of the modulus "$|...|$" involved in the definition of the $I_{i,j}$. After that $I(T)$ decreases asymptotically to zero, i.e. $I(T)$ cannot exhibit SDE. It would be more convenient to find the interaction time after which $I(T) = 0$. By means of (24)-(25), one can easily prove that the solution of this equation is $\exp(-\Gamma t) = 0$. This means that $I(T)$ needs infinite time to vanish. From this example and the above one, one can see that it is impossible to find sudden
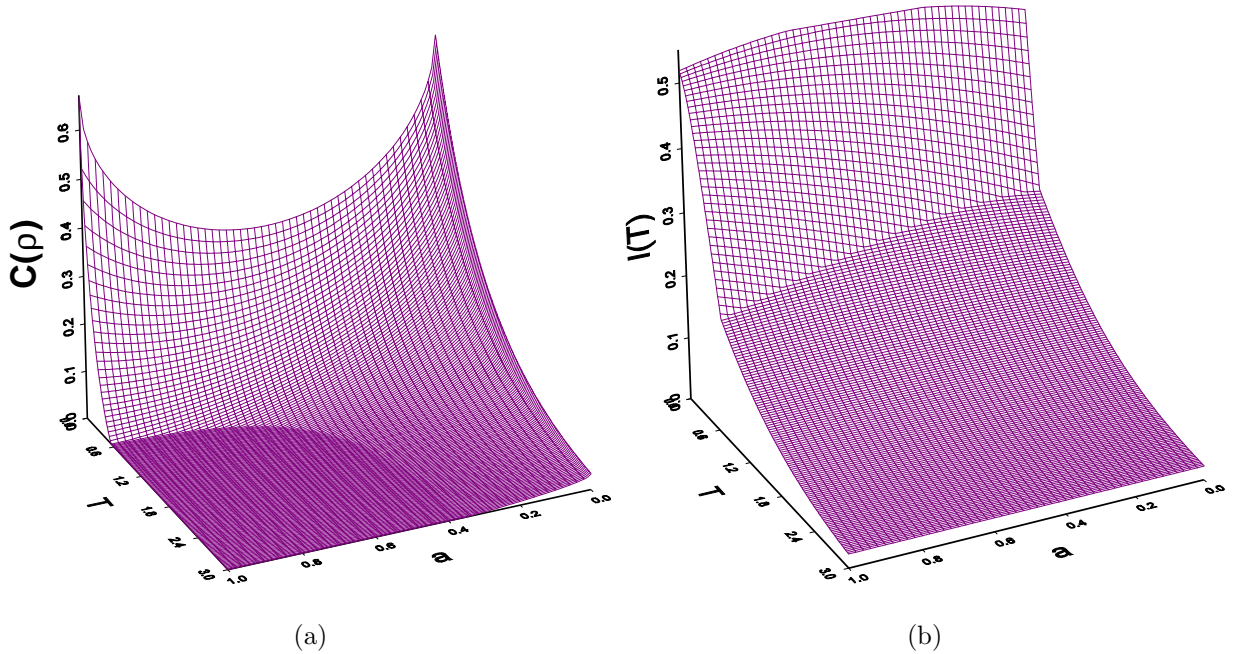

FIG. 2: Atom-atom entanglement quantified by $C(\rho)$ (a) and I (b) when the atoms are initially prepared in the mixed entangled state with $b_{11} = a/3 \geq 0, b_{22} = b_{33} = b_{23} = b_{32} = 1/3, \quad b_{44} = (1-a)/3$.

death of correlation/entanglement in the framework of the I. The reason for this is obvious: I carries information on both classical and nonclassical correlation. More illustratively, SDE occurs when the entanglement evolves abruptly to the classical one, however, the latter is still noticeable in I. It is worthwhile mentioning that this behavior, the disappearance of SDE, has already been addressed also via quantum discord for dissipative two-qubit dynamics within Markovian and non-Markovian environments [9, 10]. Choosing initial conditions that manifest SDE, the authors there have compared the dynamics of entanglement with that of quantum discord. The analysis showed that in all cases where the entanglement, in terms

of the concurrence, suddenly disappears during a finite time interval, the quantum discord vanishes only in the asymptotic limit or at discrete instants. This reveals that quantum discord is more robust than the concurrence against SDE.

It is imperative to extend the definition of the bipartite indicator I to multipartite systems. Indeed, in the multipartite case, apart from fully separable and fully correlated/entangled states, there also exists the notion of partial separability. Here, we focus our discussion on full correlation/entanglement, however, the partial one can be treated similarly. For multipartite systems, one has to deal with all permutations of the correlation elements of the spin operators. For instance, for the tripartite, one should consider the correlation elements of both two particles and three particles. However, this cannot be done so easily. Alternatively, one may quantify the correlation for each two qubits individually using (8) and eventually carry out the average over all indicators. For fully correlated/entangled states, this has to be done under the condition that the correlation must be available in each pair, i.e. none of the individual indicators equals zero. In this approach one usually exploits the pairwise symmetry of multipartite system, which simplifies the calculation of these indicators considerably. In order to verify this technique, it is sufficient to compute the correlation of the three particle GHZ (10), which reads:

$$I = \frac{1}{3}[I^{(1,2)} + I^{(2,3)} + I^{(1,3)}] = 1, \tag{27}$$

where the indicator $I^{(i,j)}$ denotes the correlation between particle i and particle j, and we have used the results of the example (1). The result given by (27) represents nonclassical correlation and this agrees with the fact that the GHZ is a maximally entangled tripartite state. Nonetheless, this result has already been deduced in terms of the 3-tangle τ_3 (or residual tangle) [22], however, τ_3 vanishes for a particular class of entangled three-qubit states [23]. This class is represented by the state $|W\rangle$:

$$|W\rangle = \frac{1}{\sqrt{3}}[|e_1, g_2, g_3\rangle + |g_1, e_2, g_3\rangle + |g_1, g_2, e_3\rangle]. \tag{28}$$

By applying our criterion to (28), we obtain $I = 5/9$, i.e. the W state is entangled but not maximally. This shows that the GHZ and the W state are locally inequivalent.

In conclusion, we have developed a new criterion for quantifying correlation and/or entanglement of the bipartite systems. The proposed measure is simple and involves only calculation of average values of the spin matrices, without the need for operations on the

density matrix of the system nor the calculation of eigenvalues, as it happens with the concurrence. Also we have explained how it can be extended to treat the correlation of multipartite systems. The criterion is based on observable quantities, which have a clear physical meaning. This can give reliable physical interpretation for entanglement. Also it suggests a way for measuring the correlation experimentally. We have compared the results obtained from our criterion with those from the concurrence, and for particular examples with the degree of separability. There is agreement and disagreement between the behaviors of both criteria. Generally, there is agreement among the criteria in the entanglement domain. The $C(\rho)$ and I could have similar behavior when $C(\rho)$ behaves smoothly. Moreover, from the comparison with the other criteria, we have found, when $I > 1/3$, the bipartite is entangled, however, for $I \leq 1/3$ the bipartite may be correlated classically or nonclassically. Also, we have shown that the indicator I cannot exhibit SDE. In this regard, the behavior of I is quite similar to that of quantum discord [9, 10]. Now, the question: is there any relationship between I and quantum discord? This point will be studied in a future work where more general Hamiltonians and initial conditions will be considered. From the analysis given in this Letter one can arrive at the following fact: if it is proved that a physical system is entangled in terms of any criterion, it will be automatically entangled in the framework of I, i.e. $I \neq 0$, however, the reverse is not true. Finally, we believe that the criterion given in this Letter may have a wide range of potential applications in quantum information theory. For instance, the experimentalists always measure the correlation in the quantum systems regardless it is classical or nonclassical , since they cannot distinguish between them. In this sense, the new criterion could be of interest.

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